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**LETTER TO THE EDITOR**

# The pair connectedness for directed percolation on the honeycomb and diamond lattices

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**Abstract.** The pair connectedness for directed site percolation on the honeycomb and diamond lattices is related to that of the square and simple cubic lattices respectively. In the case of bond percolation the same correspondence leads to site–bond percolation on the latter pair of lattices.

Directed percolation first introduced by Broadbent and Hammersley (1957) has recently proved to be of considerable interest. Dhar *et al* (1982) have obtained the relation

$$xG_2^{\text{hon}}(x, y) = G^{\text{sq}}(x^2, y+xy) \tag{1}$$

between the generating functions for site animals on the honeycomb and square lattices, from which they deduce that the critical probabilities for directed site percolation are related by

$$p_{\text{c}}^{\text{hon}} = (p_{\text{c}}^{\text{sq}})^{1/2}. \tag{2}$$

Equation (2) follows from (1) since in general $G(p, 1-p) = p(1-P(p))$ where $P(p)$ is the percolation probability and hence

$$P^{\text{hon}}(p) = P^{\text{sq}}(p^2). \tag{3}$$

The partial derivatives $G_x(p, 1-p)$ and $G_y(p, 1-p)$ determine the mean size and mean perimeter of clusters in the percolation problem, and taking the $x$ derivative of (1) expresses the mean size for the honeycomb lattice in terms of the mean size and mean perimeter of the square lattice. Here we show that an approach using the pair connectedness leads to a direct relation between the mean size functions. We shall also consider the spatial moments which determine the two connectedness lengths of this model (Kinzel and Yeomans 1981, Essam and De'Bell 1981).

The results have an immediate extension to site percolation on the diamond lattice. In the case of bond percolation the situation is not so simple, and to obtain a correspondence it is necessary to consider percolation models on the square and cubic lattices in which both sites and bonds are randomly occupied.

Suppose that the honeycomb lattice $H$ is directed and coloured as in Dhar *et al* (1982, figure 1($c$)). In the discussion of connectedness lengths below, the geometry as well as the topology of the lattice is important, and we suppose that the nearest-neighbour distance $a = 2$ units and that the bond angles are 120°. The vertices on the two triangular sublattices $T_1$ (black) and $T_2$ (white) have one and two bonds directed away from them respectively. We first relate the pair connectedness of $H$ to that of a lattice $SQ$ which is topologically a directed square lattice. The latter may be obtained from $H$ by contracting the bonds which are directed from $T_1$ to $T_2$. During the contraction the sites of $T_1$ are fixed and $T_2$ approaches $T_1$, and thus $SQ$ has the sites of $T_1$ and its lattice parameter is $2\sqrt{3}$ (see figure 1). The pair connectedness $P_i^{\text{hon}}(p, j)$ is the probability that there is at least one path of occupied sites from a chosen site $\rho_i$ of $T_i$ (figure 1) to the site $j$ of $H$, given that $\rho_i$ is occupied.

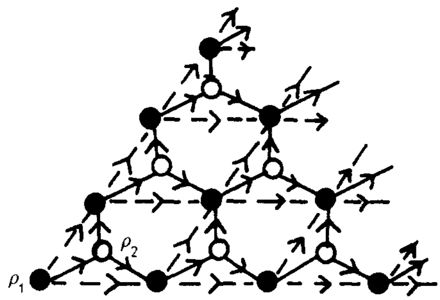


**Figure 1.** Sublattices $T_1$ (●) and $T_2$ (○) of the honeycomb lattice. Contraction of bonds from $T_1$ to $T_2$ gives the directed square lattice.

Let $\pi_{ij}$ be the set of all possible paths from $\rho_i$ to $j$. By inclusion and exclusion

$$P_i^{\text{hon}}(p, j) = \sum_{\phi \subset \pi'_{ij} \subseteq \pi_{ij}} (-1)^{n_{ij}} P^{\text{hon}}(p, \pi'_{ij}) \tag{4}$$

where $P(p, \pi'_{ij})$ is the probability that the $n_{ij}$ occupied paths $\pi'_{ij}$ occur simultaneously. For $i = 2$ and $j \in T_2$ the above contraction induces a natural correspondence between paths on $H$ and paths on $SQ$. If in the contraction $\pi'_{2j} \to \bar{\pi}'_j$ then, since $\rho_2 \to \rho_1$ and there is a two-to-one correspondence between the other sites of $H$ used by $\pi'_{2j}$ and $\bar{\pi}'_j$, we find that

$$P^{\text{hon}}(p, \pi'_{2j}) = P^{\text{sq}}(p^2, \bar{\pi}'_j). \tag{5}$$

Hence

$$P_2^{\text{hon}}(p, j) = P^{\text{sq}}(p^2, j'), \qquad j \in T_2, \tag{6}$$

where $j'$ is the site of $T_1$ approached by $j$ in the contraction. Similarly

$$P_2^{\text{hon}}(p, j) = p^{-1} P^{\text{sq}}(p^2, j), \qquad j \in T_1, \tag{7}$$

and since

$$P_1^{\text{hon}}(p, j) = p P_2^{\text{hon}}(p, j), \qquad j \in H, \tag{8}$$

we have

$$P_1^{\text{hon}}(p, j) = P^{\text{sq}}(p^2, j), \qquad j \in T_1, \tag{9}$$

and

$$P_1^{\text{hon}}(p, j) = p P^{\text{sq}}(p^2, j'), \qquad j \in T_2. \tag{10}$$

The moments of the pair connectedness are defined by

$$\mu_{lm}^{\text{hon}}(p,i)=\sum_{j\in H} x_{ij}^{l} t_{ij}^{m} P_i^{\text{hon}}(p,j) \tag{11}$$

where $x_{ij}$, $t_{ij}$ are the coordinates of $j$ relative to axes at $\rho_i$ perpendicular and parallel to $\rho_1-\rho_2$ respectively. In particular, $\mu_{00}^{\text{hon}}(p,i)$ is the mean size of clusters rooted at $\rho_i$. These moments may be related to

$$\mu_{lm}^{\text{sq}}(p)=\sum_{j\in SQ} x_j^{l} t_j^{m} P^{\text{sq}}(p,j) \tag{12}$$

where $x_j$ and $t_j$ are the coordinates of $j$ on the directed square lattice with unit lattice parameter and 90° bond angles. Using

$$x_{ij}=\begin{cases}6^{1/2}x_j, & j\in T_1\\ 6^{1/2}x_{j'}, & j\in T_2,\end{cases}\qquad t_{ij}=2^{1/2}\begin{cases}3t_j\quad , & i=1, j\in T_1,\\ 3t_{j'}+2^{1/2}, & i=1, j\in T_2,\\ 3t_j-2^{1/2}, & i=2, j\in T_1,\\ 3t_{j'}\quad . & i=2, j\in T_2,\end{cases} \tag{13}$$

we find

$$\mu_{lm}^{\text{hon}}(p,1)=\sum_{j\in SQ}(6x_j)^{l/2}2^{m/2}P^{\text{sq}}(p^2,j)[(3t_j)^m+p(3t_j+2^{1/2})^m] \tag{14}$$

and

$$\mu_{lm}^{\text{hon}}(p,2)=\sum_{j\in SQ}{}' (6x_j)^{l/2}2^{m/2}P^{\text{sq}}(p^2,j)[(3t_j)^m+p^{-1}(3t_j-2^{1/2})^m] \tag{15}$$

where the prime means that $j=\rho_1$ is excluded from the sum. The case $m=0$ gives the simple result

$$\mu_{l0}^{\text{hon}}(p,1)=p\mu_{l0}^{\text{hon}}(p,2)=6^{l/2}(1+p)\mu_{l0}^{\text{sq}}(p^2) \tag{16}$$

and hence the transverse connectedness lengths on the two lattices are related by the factor $6^{1/2}$. For $l\neq 0$ the even part of $\mu_{lm}^{\text{hon}}$ is proportional to $\mu_{lm}^{\text{sq}}(p^2)$ but the odd part depends on $\mu_{lm'}^{\text{sq}}(p^2)$ for $m'\leq m$. Thus for $l=0$, $m=2$

$$\mu_{02}^{\text{hon}}(p,2)=18\mu_{02}^{\text{sq}}(p^2)+p^{-1}[18\mu_{02}^{\text{sq}}(p^2)-12\times 2^{1/2}\mu_{01}^{\text{sq}}(p^2)+4(\mu_{00}^{\text{sq}}(p^2)-1)]. \tag{17}$$

The longitudinal connectedness-lengths are not simply related but as $p\to p_c^{\text{hon}}$, $\xi_{\parallel}^{\text{hon}}(p)\sim 3\times 2^{1/2}\xi_{\parallel}^{\text{sq}}(p^2)$.

Extension of the above arguments to three-dimensional site percolation shows that (2), (6)–(10) are valid with the honeycomb replaced by the diamond lattice (with $a=3$) and the square by the simple cubic lattice (with $a=1$). Equations (14) and (15) become

$$\mu_{lm}^{\text{dia}}(p,1)=\sum_{j\in SC}(12x_j)^{l/2}3^{m/2}P^{\text{sc}}(p^2,j)[(4t_j)^m+p(4t_j+3^{1/2})^m] \tag{18}$$

and

$$\mu_{lm}^{\text{dia}}(p,2)=\sum_{j\in SC}{}'(12x_j)^{l/2}3^{m/2}P^{\text{sc}}(p^2,j)[(4t_j)^m+p^{-1}(4t_j-3^{1/2})^m]. \tag{19}$$

For $l=0$ equation (16) is valid with 6 replaced by 12, which implies that

$$p_c^{dia} = (p_c^{sc})^{1/2}. \qquad (20)$$

Extension to higher-dimensional lattices is clearly possible.

The correspondence between paths described above is also valid for bond percolation, however, the factors $p$ are now associated with the bonds rather than the sites. Thus, on contraction of the bonds directed from $T_1$ to $T_2$, the factors associated with these bonds become associated with the sites onto which they are contracted. The other bonds of $H$ correspond one-to-one with the bonds of $SQ$. The honeycomb directed bond problem therefore corresponds to percolation on the directed square lattice with bonds and sites having equal probabilities of being present. Similar remarks apply to higher-dimensional lattices.

Finally, it is clear that there is a correspondence between directed site–bond percolation with general parameters $p_s$, $p_b$ on the above pairs of lattices. Thus

$$p_s^{sq} = (p_s^{hon})^2 p_b^{hon}, \qquad p_b^{sq} = p_b^{hon}, \qquad (21)$$

which provides a mapping between the critical curves.

Finally, we note that the value of $p_c = 0.8228 \pm 0.0001$ for the bond–site problem on the square lattice obtained by Kinzel and Yeomans is in excellent agreement with the value of $P_c = 0.8226 \pm 0.0002$ obtained by Blease (1977) for the bond problem on the directed honeycomb lattice.

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